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Fuzzy ideal topological spaces

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Abstract. In this paper, it is introduced the notion of r-fuzzy ideal separation axioms T_i, $i = 0, 1, 2$ based on a fuzzy ideal $\mathcal{I}$ on a fuzzy topological space (X, τ). An r-fuzzy ideal connectedness related to the fuzzy ideal $\mathcal{I}$ is introduced which has relations with a previous r-fuzzy connectedness. An r-fuzzy ideal compactness related to $\mathcal{I}$ is introduced which has also relations with many other types of fuzzy compactness.

Keywords: fuzzy ideal, fuzzy separation axioms, fuzzy compactness, fuzzy connectedness

1. Introduction and Preliminaries

This is a way to use a fuzzy ideal $\mathcal{I}$ defined on a fuzzy topological space (X, τ) giving generalizations of many notions and results in fuzzy topological spaces. r-fuzzy ideal $T_i,\ i = 0, 1, 2$ separation axioms are new types of fuzzy separation axioms related with the fuzzy ideal $\mathcal{I}$ on X. It is proved many implications between these r-fuzzy ideal $T_i,\ i = 0, 1, 2$ spaces and the previous r-fuzzy $T_i,\ i = 0, 1, 2$ defined in [7] and studied in [4–6], and also the preimage and the image of r-fuzzy ideal $T_i,\ i = 0, 1, 2$ spaces are r-fuzzy ideal $T_i,\ i = 0, 1, 2$ spaces as well. r-fuzzy ideal connectedness is introduced related with $\mathcal{I}$ giving a generalization of the r-fuzzy connectedness notion ([9, 10]). The image of r-fuzzy ideal connected is r-fuzzy ideal connected as well. r-fuzzy ideal compactness is introduced using the fuzzy ideal $\mathcal{I}$ on X giving a generalization of many other fuzzy compactness notions [1, 11]. The image of r-fuzzy compact is r-fuzzy ideal compact, and many special cases are deduced.

Note that: In [3], the author used the ideal notion to reduce the soft boundary region in ordinary soft rough topological space but here we joined the fuzzy ideal notion to fuzzy topology in sense of Šostak without concerning soft roughness. In [12], the authors introduced fuzzy soft separation axioms and fuzzy soft connectedness for fuzzy soft topological spaces in sense of Chang but in this paper we used the fuzzy ideal notion in defining fuzzy ideal separation axioms, fuzzy ideal connectedness and fuzzy ideal compactness in sense of Šostak.

Throughout the paper, X refers to an initial universe, I^X is the set of all fuzzy sets on X (where $I = [0, 1]$, $I_0 = (0, 1]$, $\lambda^c(x) = 1 - \lambda(x)\, \forall x \in X$ and for all $t \in I$, $\bar{t}(x) = t\ \forall x \in X$). A fuzzy point x_t is defined by $x_t(y) = t$ at $y = x$ and $x_t(y) = 0$ otherwise.

(X, τ) is a fuzzy topological space ([14]), if $\tau : I^X \to I$ satisfies the following conditions:

(O1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,
(O2) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ for all $\lambda_1, \lambda_2 \in I^X$,
(O3) $\tau(\bigvee_{j \in J} \lambda_j) \geq \bigwedge_{j \in J} \tau(\lambda_j)$ for all $\{\lambda_j\}_{j \in J} \subseteq I^X$.

A map $\mathcal{I} : I^X \to I$ is called a fuzzy ideal ([13]) on X if it satisfies:

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(1) $\mathcal{I}(\overline{0}) = 1$,
(2) $\lambda \leq \mu \ \Rightarrow\ \mathcal{I}(\lambda) \geq \mathcal{I}(\mu)$ for all $\lambda, \mu \in I^X$,
(3) $\mathcal{I}(\lambda \vee \mu) \geq \mathcal{I}(\lambda) \wedge \mathcal{I}(\mu)$ for all $\lambda, \mu \in I^X$.

If $\mathcal{I}_1$ and $\mathcal{I}_2$ are fuzzy ideals on X, we have $\mathcal{I}_1$ is finer than $\mathcal{I}_2$ ($\mathcal{I}_2$ is coarser than $\mathcal{I}_1$), denoted by $\mathcal{I}_1 \leq \mathcal{I}_2$ iff $\mathcal{I}_1(\lambda) \leq \mathcal{I}_2(\lambda)\ \forall \lambda \in I^X$. The triple $(X, \tau, \mathcal{I})$ is called a fuzzy ideal topological space.

Define the fuzzy ideal $\mathcal{I}^\circ$ by

$$\mathcal{I}^\circ(\mu) = \begin{cases} 1 \text{ at } \mu = \overline{0}, \\ 0 \text{ otherwise} \end{cases}$$

Recall that the fuzzy difference between two fuzzy sets is defined as follows ([8]):

$$(\lambda \bar{\wedge} \mu) = \begin{cases} \overline{0} & \text{if } \lambda \leq \mu, \\ \lambda \wedge \mu^c & \text{otherwise.} \end{cases}$$

Definition 1. [8] Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\lambda \in I^X$. Then, the r-fuzzy open local function $\lambda^*_r(\tau, \mathcal{I})$ of λ is defined by

$$\lambda^*_r(\tau, \mathcal{I}) = \bigwedge\{\mu \in I^X : \mathcal{I}(\lambda \bar{\wedge} \mu) \geq r,\ \tau(\mu^c) \geq r\}.$$

Occasionally, we will write λ^*_r or $\lambda^*_r(\mathcal{I})$ for $\lambda^*_r(\tau, \mathcal{I})$ and it will be no ambiguity.

If $\mathcal{I} = \mathcal{I}^\circ$ then, for each $\lambda \in I^X$, $r \in I_0$, we have $\lambda^*_r = \mathrm{cl}_\tau(\lambda, r)$ ([8]).

Proposition 1. [8] *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\mathcal{I}_1, \mathcal{I}_2$ be fuzzy ideals on X. Then,*

(1) $\lambda \leq \mu$ *implies* $\lambda^*_r \leq \mu^*_r$.
(2) *If* $\mathcal{I}_1 \leq \mathcal{I}_2$, *then* $\lambda^*_r(\mathcal{I}_1) \geq \lambda^*_r(\mathcal{I}_2)$.
(3) $\lambda^*_r = \mathrm{cl}_\tau(\lambda^*_r, r) \leq \mathrm{cl}_\tau(\lambda, r)$ *and* $(\lambda^*_r)^*_r \leq \lambda^*_r$.
(4) $\lambda^*_r \vee \mu^*_r \leq (\lambda \vee \mu)^*_r$, *and* $\lambda^*_r \wedge \mu^*_r \geq (\lambda \wedge \mu)^*_r$.

In the following example, it is shown that: $\lambda^*_r \not\leq (\lambda^*_r)^*_r$.

Example 1. Let τ be a fuzzy topology and $\mathcal{I}$ a fuzzy ideal defined on X such that

$$\tau(\lambda) = \begin{cases} 1 \text{ at } \lambda = \overline{0}, \overline{1} \\ \frac{1}{2} \text{ at } \lambda = \overline{0.5} \\ \frac{1}{2} \text{ at } \lambda = \overline{0.6} \\ 0 \text{ otherwise,} \end{cases}$$

$$\mathcal{I}(\lambda) = \begin{cases} 1 \text{ at } \lambda = \overline{0} \\ \frac{1}{2} \text{ at } \lambda = \overline{0.5} \\ \frac{2}{3} \text{ at } \overline{0} < \lambda < \overline{0.5} \\ 0 \text{ otherwise.} \end{cases}$$

$$(\overline{0.6})^*_{\frac{1}{2}} = \bigwedge\{\mu : \mathcal{I}(\overline{0.6} \bar{\wedge} \mu) \geq \frac{1}{2}, \tau(\mu^c) \geq \frac{1}{2}\} = \overline{0.5},$$

but $((\overline{0.6})^*_{\frac{1}{2}})^*_{\frac{1}{2}} = (\overline{0.5})^*_{\frac{1}{2}} = \bigwedge\{\mu : \mathcal{I}(\overline{0.5} \bar{\wedge} \mu) \geq \frac{1}{2},\ \tau(\mu^c) \geq \frac{1}{2}\} = \overline{0}$.

Proposition 2. [8] *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\{\mu_j : j \in J\} \subseteq I^X$ a family. Then,*

(1) $\bigvee((\mu_j)^*_r : j \in J) \leq (\bigvee(\mu_j) : j \in J)^*_r$.
(2) $\bigwedge((\mu_j)^*_r : j \in J) \geq (\bigwedge(\mu_j) : j \in J)^*_r$.

Definition 2. [8] Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\mu \in I^X$. Then,

$$\mathrm{cl}^*_\tau(\mu, r) = \mu \vee \mu^*_r \quad \text{and} \quad \mathrm{int}^*_\tau(\mu, r) = \mu \wedge ((\mu^c)^*_r)^c.$$

If $\mathcal{I} = \mathcal{I}^\circ$, then for each $\mu \in I^X$, $r \in I_0$, $\mathrm{cl}^*_\tau(\mu, r) = \mu \vee \mu^*_r = \mu \vee \mathrm{cl}_\tau(\mu, r) = \mathrm{cl}_\tau(\mu, r)$.

Proposition 3. [8] *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\lambda, \mu \in I^X$, $r \in I_0$. Then,*

(1) $\mathrm{int}^*_\tau(\lambda \vee \mu, r) \geq \mathrm{int}^*_\tau(\lambda, r) \vee \mathrm{int}^*_\tau(\mu, r)$.
(2) $\mathrm{int}_\tau(\lambda, r) \leq \mathrm{int}^*_\tau(\lambda, r) \leq \lambda \leq \mathrm{cl}^*_\tau(\lambda, r) \leq \mathrm{cl}_\tau(\lambda, r)$.
(3) $\mathrm{cl}^*_\tau(\lambda^c, r) = (\mathrm{int}^*_\tau(\lambda, r))^c$ and $\mathrm{int}^*_\tau(\lambda^c, r) = (\mathrm{cl}^*_\tau(\lambda, r))^c$.
(4) $\mathrm{int}^*_\tau(\lambda \wedge \mu, r) \leq \mathrm{int}^*_\tau(\lambda, r) \wedge \mathrm{int}^*_\tau(\mu, r)$.

Corollary 1. [8] *Let $(X, \tau_1, \mathcal{I})$, $(X, \tau_2, \mathcal{I})$ be fuzzy ideal topological spaces and $\tau_1 \leq \tau_2$. Then, for each $\lambda \in I^X$, $r \in I_0$, we have $\lambda^*_r(\tau_2, \mathcal{I}) \leq \lambda^*_r(\tau_1, \mathcal{I})$.*

Corollary 2. [8] *Let $(X, \tau, \mathcal{I}_1)$, $(X, \tau, \mathcal{I}_2)$ be fuzzy ideal topological spaces and $\mathcal{I}_1 \leq \mathcal{I}_2$. Then, for each $\lambda \in I^X$, $r \in I_0$, we have $\lambda^*_r(\tau, \mathcal{I}_1) \geq \lambda^*_r(\tau, \mathcal{I}_2)$.*

Proposition 4. [8] *Let (X, τ) be a fuzzy topological space and $\mathcal{I}_1$, $\mathcal{I}_2$ fuzzy ideals on X. Then, for each $\lambda \in I^X$, $r \in I_0$, we have $\lambda^*_r(\tau, \mathcal{I}_1 \wedge \mathcal{I}_2) = \lambda^*_r(\tau, \mathcal{I}_1) \vee \lambda^*_r(\tau, \mathcal{I}_2)$.*

2. Fuzzy ideal r-(t, s)-T_i separation axioms

Here, we introduce fuzzy separation axioms in fuzzy ideal topological spaces.

Definition 3.

(1) A fuzzy ideal topological space $(X, \tau, \mathcal{I})$ is called r-(t, s)-FI-T_0 if for $t, s \in I_0$, then $x \neq y$ in X implies that there exists $\lambda \in I^X, r \in I_0$ with $t \leq \mathrm{int}^*_\tau(\lambda, r)(x)$ such that $t > \lambda(y)$ or there exists $\mu \in I^X, r \in I_0$ with $s \leq \mathrm{int}^*_\tau(\mu, r)(y)$ such that $s > \mu(x)$.

(2) A fuzzy ideal topological space $(X, \tau, \mathcal{I})$ is called r-(t, s)-FI-T_1 if for $t, s \in I_0$, then $x \neq y$ in X implies that there exist $\lambda, \mu \in I^X$, $r \in I_0$ with $t \leq \mathrm{int}^*_\tau(\lambda, r)(x)$, $s \leq \mathrm{int}^*_\tau(\mu, r)(y)$ such that $t > \lambda(y)$ and $s > \mu(x)$.

(3) A fuzzy ideal topological space $(X, \tau, \mathcal{I})$ is called r-(t, s)-FI-T_2 if for $t, s \in I_0$, then $x \neq y$ in X implies that there exist $\lambda, \mu \in I^X$, $r \in I_0$ with $t \leq \mathrm{int}^*_\tau(\lambda, r)(x)$, $s \leq \mathrm{int}^*_\tau(\mu, r)(y)$ such that $(t \wedge s) > \sup(\lambda \wedge \mu)$.

Remark 1. Consider a fuzzy ideal topological space $(X, \tau, \mathcal{I})$ with $\mathcal{I} = \mathcal{I}^\circ$. Then, the graded fuzzy separation axioms defined in [7] and the r-(t, s)-FI-T_i separation axioms are identical, $i = 0, 1, 2$.

Any fuzzy topological space (X, τ) satisfying (t, s)-T_i separation axiom as defined in [7] will be r-(t, s)-FI-T_i with respect to some fuzzy ideal on X as well but not converse, $i = 0, 1, 2$. It is coming from that: $\mathrm{int}_\tau(\mu, r) \leq \mathrm{int}^*_\tau(\mu, r)\ \forall \mu \in I^X, r \in I_0$.

Proposition 5. *Every r-(t, s)-FI-T_i fuzzy ideal topological space $(X, \tau, \mathcal{I})$ is an r-(t, s)-FI-T_{i-1} space, $i = 1, 2$.*

Proof. r-(t, s)-FI-$T_2 \Rightarrow r$-(t, s)-FI-T_1: Let $(X, \tau, \mathcal{I})$ be an r-(t, s)-FI-T_2 space, and suppose that $(X, \tau, \mathcal{I})$ is not r-(t, s)-FI-T_1. That is, for all $x \neq y$ in X and for all $\lambda \in I^X, r \in I_0$ with $t \leq \mathrm{int}^*_\tau(\lambda, r)(x)$, suppose that $\lambda(y) \geq t; t \in I_0$. Now, for $\mu \in I^X$ with $s \leq \mathrm{int}^*_\tau(\mu, r)(y)$; $s \in I_0$, we get that $s \leq \mu(y)$, and thus $\sup(\lambda \wedge \mu) \geq (\lambda \wedge \mu)(y) \geq (t \wedge s)$, which is a contradiction to $(X, \tau, \mathcal{I})$ is an r-(t, s)-FI-T_2 space. Hence, $(X, \tau, \mathcal{I})$ is an r-(t, s)-FI-T_1 space.

r-(t, s)-FI-$T_1 \Rightarrow r$-(t, s)-FI-T_0: It is clear. □

Recall that: a mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy continuous ([14]) if

$$\mathrm{int}_\tau(f^{-1}(\nu), r) \geq f^{-1}(\mathrm{int}_\sigma(\nu, r))\ \forall \nu \in I^Y, r \in I_0.$$

It is equivalent to satisfy the following

$$\mathrm{cl}_\tau(f^{-1}(\nu), r) \leq f^{-1}(\mathrm{cl}_\sigma(\nu, r))\ \forall \nu \in I^Y, r \in I_0.$$

Now, let us call a mapping $f : (X, \tau) \to (Y, \sigma, \mathcal{I}')$ fuzzy ideal continuous provided that

$$\mathrm{int}_\tau(f^{-1}(\nu), r) \geq f^{-1}(\mathrm{int}^*_\sigma(\nu, r))\ \forall \nu \in I^Y, r \in I_0.$$

It is easily shown that it is equivalent to

(IC) $\mathrm{cl}_\tau(f^{-1}(\nu), r) \leq f^{-1}(\mathrm{cl}^*_\sigma(\nu, r)) \quad \forall \nu \in I^Y, r \in I_0$.

Also, let us call $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ a fuzzy ideal open mapping provided that

$$f(\mathrm{int}^*_\tau(\lambda, r)) \leq \mathrm{int}_\sigma(f(\lambda), r)\ \forall \lambda \in I^X, r \in I_0.$$

It is easily shown that it is equivalent to

(IO) $\mathrm{cl}_\sigma(f(\lambda), r) \leq f(\mathrm{cl}^*_\tau(\lambda, r)) \quad \forall \lambda \in I^X, r \in I_0$.

It is clear that: any map f satisfying condition (IC) (or fuzzy ideal continuous) will be a fuzzy continuous mapping $f : (X, \tau) \to (Y, \sigma)$, but not every fuzzy continuous mapping $f : (X, \tau) \to (Y, \sigma)$ will satisfy the condition (IC) (or fuzzy ideal continuous) with respect to a fuzzy ideal $\mathcal{I}'$ on Y. Also, a map f satisfying condition (IO) (or fuzzy ideal open) will be a fuzzy open mapping $f : (X, \tau) \to (Y, \sigma)$, but not every fuzzy open mapping $f : (X, \tau) \to (Y, \sigma)$ will satisfy the condition (IO) (or fuzzy ideal open) with respect to a fuzzy ideal $\mathcal{I}$ on X.

Theorem 1. *Let $(X, \tau, \mathcal{I}), (Y, \sigma, \mathcal{I}')$ be fuzzy ideal topological spaces and $f : (X, \tau) \to (Y, \sigma, \mathcal{I}')$ be an injective fuzzy ideal continuous mapping. Then, $(X, \tau, \mathcal{I})$ is an r-(t, s)-FI-T_i space if $(Y, \sigma, \mathcal{I}')$ is an r-(t, s)-FI-T_i space, $i = 0, 1, 2$.*

Proof. Since $x \neq y$ in X implies that $f(x) \neq f(y)$ in Y and for $(Y, \sigma, \mathcal{I}')$ is r-(t, s)-FI-T_2, then there exist $\nu, \rho \in I^Y$, $r \in I_0$ with $t \leq \mathrm{int}^*_\sigma(\nu, r)(f(x))$, $s \leq \mathrm{int}^*_\sigma(\rho, r)(f(y))$; $t, s \in I_0$ so that $t \wedge s > \sup(\nu \wedge \rho)$, that is, $t \leq f^{-1}(\mathrm{int}^*_\sigma(\nu, r))(x)$, $s \leq f^{-1}(\mathrm{int}^*_\sigma(\rho, r))(y)$; $t, s \in I_0$. Since f is fuzzy ideal continuous, $t \leq \mathrm{int}_\tau(f^{-1}(\nu), r)(x) \leq \mathrm{int}^*_\tau(f^{-1}(\nu), r)(x)$, $s \leq \mathrm{int}_\tau(f^{-1}(\rho), r)(y) \leq \mathrm{int}^*_\tau(f^{-1}(\rho), r)(y)$; $t, s \in I_0$. That is, there exist $\lambda = f^{-1}(\nu), \mu = f^{-1}(\rho) \in I^X$ with $t \leq \mathrm{int}^*_\tau(\lambda, r)(x)$, $s \leq \mathrm{int}^*_\tau(\mu, r)(y)$; $t, s \in I_0$. Now, f is injective implies that $\sup(\lambda \wedge \mu) = \sup(f^{-1}(\nu) \wedge f^{-1}(\rho)) \leq \sup(\nu \wedge \rho) < t \wedge s$. Hence, $(X, \tau, \mathcal{I})$ is an r-(t, s)-FI-T_2 space.

For the cases of $(X, \tau, \mathcal{I})$ is r-(t, s)-FI-T_0 and r-(t, s)-FI-T_1, it is similar. □

Theorem 2. *Let $(X, \tau, \mathcal{I}), (Y, \sigma, \mathcal{I}^{'})$ be fuzzy ideal topological spaces and $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a surjective fuzzy ideal open mapping. Then, $(Y, \sigma, \mathcal{I}^{'})$ is an r-(t, s)-FI-T_i space if $(X, \tau, \mathcal{I})$ is an r-(t, s)-FI-T_i space, $i = 0, 1, 2$.*

Proof. Since $p \neq q$ in Y implies that there are $x \neq y$ in X where $x \in f^{-1}(p)$, $y \in f^{-1}(q)$ and for $(X, \tau, \mathcal{I})$ is r-(t, s)-FI-T_2, then there exist $\lambda, \mu \in I^X$, $r \in I_0$ with $t \leq \mathrm{int}^*_\tau(\lambda, r)(x)$, $s \leq \mathrm{int}^*_\tau(\mu, r)(y)$; $t, s \in I_0$ so that $t \wedge s > \sup(\lambda \wedge \mu)$. Now, $t \leq \bigvee_{x \in f^{-1}(p)} \mathrm{int}^*_\tau(\lambda, r)(x) = f(\mathrm{int}^*_\tau(\lambda, r))(p)$, $s \leq \bigvee_{y \in f^{-1}(q)} \mathrm{int}^*_\tau(\mu, r)(y) = f(\mathrm{int}^*_\tau(\mu, r))(q)$. From f is fuzzy ideal open, then $t \leq \mathrm{int}_\sigma(f(\lambda), r)(p) \leq \mathrm{int}^*_\sigma(f(\lambda), r)(p)$, $s \leq \mathrm{int}_\sigma(f(\mu), r)(q) \leq \mathrm{int}^*_\sigma(f(\mu), r)(q)$, which means that there exist $\nu = f(\lambda), \rho = f(\mu) \in I^Y$, $r \in I_0$ with $t \leq \mathrm{int}^*_\sigma(\nu, r)(p)$, $s \leq \mathrm{int}^*_\sigma(\rho, r)(q)$; $t, s \in I_0$. Since f is surjective,

$$\sup(\nu \wedge \rho) = \sup(f(\lambda) \wedge f(\mu)) \leq \sup(\lambda \wedge \mu) < t \wedge s.$$

Hence, $(Y, \sigma, \mathcal{I}^{'})$ is an r-(t, s)-FI-T_2 space.

For the cases of $(Y, \sigma, \mathcal{I}^{'})$ is r-(t, s)-FI-T_0 and r-(t, s)-FI-T_1, it is similar. □

Example 2. Let $X = \{x, y\}$, τ be a fuzzy topology on X defined by

$$\tau(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.5 & \text{at } \lambda = x_1 \\ 0.7 & \text{at } \lambda = y_1 \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{I}$ a fuzzy ideal on X defined by

$$\mathcal{I}(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0} \\ 0.4 & \text{at } \lambda = \overline{0.6} \\ 0.7 & \text{at } \overline{0} < \lambda < \overline{0.6} \\ 0 & \text{otherwise.} \end{cases}$$

Then, $(X, \tau, \mathcal{I})$ is a 0.6-(t, s)-FI-T_i space, $i = 0, 1, 2$ but (X, τ) is not 0.6-(t, s)-F-T_i space, $i = 1, 2$ because:

For $r = 0.6$ and any $\mu \in I^X$, we have $\mu^*_{0.6} = \bigwedge\{\nu : \mathcal{I}(\mu \bar{\wedge} \nu) \geq 0.6,\ \tau(\nu^c) \geq 0.6\}$ (where ν may be $\overline{1}$, $\overline{0}$ or x_1). That is,

$$\mu^*_{0.6} = \begin{cases} \overline{0} & \text{at } \overline{0} \leq \mu < \overline{0.6} \\ x_1 & \text{at } \mu = (x_k \vee y_m),\ m < 0.6 < k \\ \overline{1} & \text{otherwise,} \end{cases}$$

which means

$$\mathrm{int}^*_\tau(\mu, 0.6) = \begin{cases} \mu & \text{at} \overline{0.4} < \mu \leq \overline{1} \\ y_q & \text{at} \mu = (x_p \vee y_q), \\ p < 0.4 < q & \\ \overline{0} & \text{otherwise} \end{cases} \tag{1}$$

while

$$\mathrm{int}_\tau(\mu, 0.6) = \begin{cases} \mu & \text{at } \mu \in \{y_1, \overline{1}\} \\ \overline{0} & \text{otherwise} \end{cases} \tag{2}$$

For $\lambda = x_{0.6} \vee y_{0.5}$, then $\mathrm{int}^*_\tau(\lambda, 0.6)(x) = \lambda(x) = 0.6$, $\lambda(y) = 0.5 < 0.6$, and for $\mu = x_{0.5} \vee y_{0.6}$, then $\mathrm{int}^*_\tau(\mu, 0.6)(y) = \mu(y) = 0.6$, $\mu(x) = 0.5 < 0.6$, which means that for $t = s = 0.6$, we get $t \leq \mathrm{int}^*_\tau(\lambda, r)(x)$, $s \leq \mathrm{int}^*_\tau(\mu, r)(y)$ such that $\lambda(y) < t$, $\mu(x) < s$. Moreover, $\sup(\lambda \wedge \mu) = 0.5 < 0.6 = (t \wedge s)$. Hence, $(X, \tau, \mathcal{I})$ is satisfying the r-(t, s)-FI-T_i axioms, $i = 0, 1, 2$, and we can not find $\lambda, \mu \in I^X$ satisfying the fuzzy r-(t, s)-T_1 or r-(t, s)-T_2 (From Equation 2.2, $\mathrm{int}_\tau(\mu, 0.6) = \overline{0}\ \forall \mu \notin \{y_1, \overline{1}\}$).

Example 3. Let $X = \{x, y\}$ and τ be a fuzzy topology on X defined by

$$\tau(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.5 & \text{at } \lambda = x_1 \\ 0.6 & \text{at } \lambda = y_1 \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{I}$ a fuzzy ideal on X defined by

$$\mathcal{I}(\lambda) = \begin{cases} 1 - m & \text{at } \lambda \leq x_m,\ 0 \leq m \leq 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) $(X, \tau, \mathcal{I})$ is a 0.6-(t, s)-FI-T_0 space but it is neither 0.6-(t, s)-FI-T_1 space nor 0.6-(t, s)-FI-T_2 space because:

For $r = 0.6$ and any choice for $\mu \in I^X$ as $\mu = (x_k \vee y_m)$, $k \leq 0.5$, $0 \leq m < 1$, we get that $\mu^c = (x_p \vee y_q)$, $p > 0.5$, $q > 0$, and then $(\mu^c)^*_r = \overline{1}$, and for $\mu = (x_k \vee y_m)$, $k > 0.5$, $0 \leq m < 1$, we get $\mu^c = (x_p \vee y_q)$, $p \leq$

0.5, $q > 0$, and then $(\mu^c)^*_r = \overline{1}$. That is, $\text{int}^*_\tau(\mu, r)(x) = 0$ for any choice of μ with $y_1 \not\leq \mu$, and thus we can not find $\lambda, \mu \in I^X$ satisfying any of the r-(t, s)-FI-T_1 axiom or the r-(t, s)-FI-T_2 axiom, while $(X, \tau, \mathcal{I})$ could be only an r-(t, s)-FI-T_0 space (by choosing $\mu = x_{0.8} \vee y_1$, then $\text{int}^*_\tau(\mu, r)(y) = \mu(y) = 1 \geq t > 0.8 = \mu(x)$; $t = 0.9$). Moreover, (X, τ) is also satisfying the fuzzy 0.6-(t, s)-T_0 axiom (taking $\mu = y_1$, then $\text{int}_\tau(\mu, r)(y) = \mu(y) = 1 \geq t > 0 = \mu(x)$; $t = 0.6$).

(2) In case of $r = 0.8$, we deduce that: (X, τ) is not fuzzy r-(t, s)-T_i, $i = 0, 1, 2$, and $(X, \tau, \mathcal{I})$ is not r-(t, s)-FI-T_i, $i = 1, 2$ while the r-(t, s)-FI-T_0 axiom is satisfied (by taking $\mu = x_{0.8} \vee y_1$).

3. Connectedness in fuzzy ideal topological spaces

Here, we introduce the r-fuzzy ideal connectedness of a fuzzy ideal topological space $(X, \tau, \mathcal{I})$.

Definition 4. Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space. Then,

(1) the fuzzy sets $\lambda, \mu \in I^X$ are called r-fuzzy ideal separated (r-FI-separated for short) if

$$\text{cl}^*_\tau(\lambda, r) \wedge \mu = \lambda \wedge \text{cl}^*_\tau(\mu, r) = \overline{0}\ ;\ r \in I_0.$$

(2) $(X, \tau, \mathcal{I})$ is called r-fuzzy ideal connected space (r-FI-connected for short) if it could not be found r-FI-separated sets $\lambda, \mu \in I^X$, $\lambda \neq \overline{0}, \mu \neq \overline{0}$ such that $\lambda \vee \mu = \overline{1}$. That is, there are no r-FI-separated sets $\lambda, \mu \in I^X$ except $\lambda = \overline{0}$ or $\mu = \overline{0}$.

Definition 5. Let $\lambda, \mu \in I^X, \lambda \neq \overline{0}, \mu \neq \overline{0}$ such that:

(1) λ, μ are r-FI-separated and $\lambda \vee \mu = \overline{1}$. Then $(X, \tau, \mathcal{I})$ is called an r-FI-disconnected space.
(2) λ, μ are r-FI-separated and $\lambda \vee \mu = \nu$. Then ν is called r-FI-disconnected fuzzy set in I^X.
(3) λ, μ are r-FI-separated and $\lambda \vee \mu = \chi_A, A \subseteq X$. Then A is called r-FI-disconnected crisp set in I^X.

Remark 2. Consider a fuzzy ideal topological space $(X, \tau, \mathcal{I})$.

Any two r-fuzzy separated sets ([9]) λ, μ in I^X are r-FI-separated as well from that: $\text{cl}^*_\tau(\nu, r) \leq \text{cl}_\tau(\nu, r)\ \forall \nu \in I^X; r \in I_0$.

That is, r-fuzzy disconnectedness ([9]) implies r-FI-disconnectedness and thus, r-FI-connectedness implies r-fuzzy connectedness ([9]).

Example 4. Let $X = \{x, y\}$. Define $\tau, \mathcal{I} : I^X \to I$ as in Example 2.1. Then:

For $0.5 < r \leq 0.7$ and for any $\mu \in I^X$, we have

$$\text{cl}^*_\tau(\mu, r) = \begin{cases} \mu & \text{at} \overline{0} \leq \mu < \overline{0.6} \\ (\mu \vee x_1) & \text{at} \mu = (x_k \vee y_m), \\ m < 0.6 < k & \\ \overline{1} & \text{otherwise.} \end{cases} \quad (3)$$

while

$$\text{cl}_\tau(\mu, r) = \begin{cases} \mu \text{ at } \mu \in \{x_1, \overline{0}\} \\ \overline{1} \text{ otherwise.} \end{cases} \quad (4)$$

So, for $0.5 < r \leq 0.7$, we can easily find $\lambda = x_{0.7}, \mu = y_{0.5} \in I^X$ with $\text{cl}^*_\tau(\lambda, r) = x_1$, $\text{cl}^*_\tau(\mu, r) = y_{0.5}$, and then $\text{cl}^*_\tau(\lambda, r) \wedge \mu = \text{cl}^*_\tau(\mu, r) \wedge \lambda = \overline{0}$, which means there are r-FI-separated sets. But for all possible choices of such r-FI-separated sets, we have $\lambda \vee \mu \neq \overline{1}$. Hence, $(X, \tau, \mathcal{I})$ is not an r-FI-disconnected space, and hence $(X, \tau, \mathcal{I})$ is an r-FI-connected space. Note that: not every r-FI-separated sets are r-fuzzy separated sets, where $\text{cl}_\tau(\lambda, 0.7) = \text{cl}_\tau(x_{0.7}, 0.7) = \text{cl}_\tau(\mu, 0.7) = \text{cl}_\tau(y_{0.5}, 0.7) = \overline{1}$, which means that $\text{cl}_\tau(\lambda, 0.7) \wedge \mu = y_{0.5}$, $\text{cl}_\tau(\mu, 0.7) \wedge \lambda = x_{0.7}$. Hence, the result in Remark 3.1 is true.

Lemma 1. *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space. Then, for any $\mu \in I^X$, $r \in I_0$ with $\tau(\mu^c) \geq r$, we get that: $\mu^*_r \leq \mu$.*

Proof. From that: $\mu^*_r = \text{cl}_\tau(\mu^*_r, r) \leq \text{cl}^*_\tau(\mu, r) \leq \text{cl}_\tau(\mu, r) = \mu$. □

Proposition 6. *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space. Then the following are equivalent.*

(1) *$(X, \tau, \mathcal{I})$ is r-FI-connected.*
(2) *$\lambda \wedge \mu = \overline{0}$, $\tau(\lambda) \geq r, \tau(\mu) \geq r$; $r \in I_0$, and $\lambda \vee \mu = \overline{1}$ imply $\lambda = \overline{0}$ or $\mu = \overline{0}$.*
(3) *$\lambda \wedge \mu = \overline{0}$, $\tau(\lambda^c) \geq r, \tau(\mu^c) \geq r$; $r \in I_0$, and $\lambda \vee \mu = \overline{1}$ imply $\lambda = \overline{0}$ or $\mu = \overline{0}$.*

Proof. (1) ⇒ (2): Let $\lambda, \mu \in I^X$ with $\tau(\lambda) \geq r, \tau(\mu) \geq r$; $r \in I_0$ such that $\lambda \wedge \mu = \overline{0}$ and $\lambda \vee \mu = \overline{1}$. Then, $\lambda = \mu^c$ and $\mu = \lambda^c$, which means (from Lemma 3.1) that $\lambda^*_r \leq \lambda$ and $\mu^*_r \leq \mu$, and then

$\mathrm{cl}^*_\tau(\lambda, r) = \lambda \vee \lambda^* = \lambda$ and $\mathrm{cl}^*_\tau(\mu, r) = \mu \vee \mu^* = \mu$, which means that $\overline{0} = \lambda \wedge \mu = \mathrm{cl}^*_\tau(\mu, r) \wedge \lambda = \mathrm{cl}^*_\tau(\lambda, r) \wedge \mu$. That is, λ, μ are r-FI-separated sets in I^X so that $\lambda \vee \mu = \overline{1}$. But $(X, \tau, \mathcal{I})$ is r-FI-connected implies that $\lambda = \overline{0}$ or $\mu = \overline{0}$.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let $\lambda, \mu \in I^X$ with $\tau(\lambda^c) \geq r, \tau(\mu^c) \geq r$; $r \in I_0$ such that $\lambda \wedge \mu = \overline{0}$ and $\lambda \vee \mu = \overline{1}$. Then, $\lambda = \mu^c$ and $\mu = \lambda^c$, and moreover $\mathrm{cl}^*_\tau(\lambda, r) = \lambda$ and $\mathrm{cl}^*_\tau(\mu, r) = \mu$, which implies that $\overline{0} = \lambda \wedge \mu = \mathrm{cl}^*_\tau(\mu, r) \wedge \lambda = \mathrm{cl}^*_\tau(\lambda, r) \wedge \mu$. That is, λ, μ are r-FI-separated sets with $\lambda \vee \mu = \overline{1}$. From (3), we have $\lambda = \overline{0}$ or $\mu = \overline{0}$. Hence, $(X, \tau, \mathcal{I})$ is an r-FI-connected space. □

Proposition 7. *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\lambda \in I^X$. Then, the following are equivalent.*

(1) *λ is r-FI-connected.*
(2) *If μ, ρ are r-FI-separated sets with $\lambda \leq \mu \vee \rho$, then $\lambda \wedge \mu = \overline{0}$ or $\lambda \wedge \rho = \overline{0}$.*
(3) *If μ, ρ are r-FI-separated sets with $\lambda \leq \mu \vee \rho$, then $\lambda \leq \mu$ or $\lambda \leq \rho$.*

Proof. (1) $\Rightarrow$ (2): Let μ, ρ be r-FI-separated with $\lambda \leq \mu \vee \rho$. That is, $\mathrm{cl}^*_\tau(\mu, r) \wedge \rho = \mathrm{cl}^*_\tau(\rho, r) \wedge \mu = \overline{0}$; $r \in I_0$ so that $\lambda \leq \mu \vee \rho$. Since $\mathrm{cl}^*_\tau(\lambda \wedge \mu, r) \leq \mathrm{cl}^*_\tau(\lambda, r) \wedge \mathrm{cl}^*_\tau(\mu, r)$ and $\mathrm{cl}^*_\tau(\lambda \wedge \rho, r) \leq \mathrm{cl}^*_\tau(\lambda, r) \wedge \mathrm{cl}^*_\tau(\rho, r)$, we get that $\mathrm{cl}^*_\tau(\lambda \wedge \mu, r) \wedge (\lambda \wedge \rho) \leq (\mathrm{cl}^*_\tau(\lambda, r) \wedge \lambda) \wedge (\mathrm{cl}^*_\tau(\mu, r) \wedge \rho) = \lambda \wedge \overline{0} = \overline{0}$, $\mathrm{cl}^*_\tau(\lambda \wedge \rho, r) \wedge (\lambda \wedge \mu) \leq (\mathrm{cl}^*_\tau(\lambda, r) \wedge \lambda) \wedge (\mathrm{cl}^*_\tau(\rho, r) \wedge \mu) = \lambda \wedge \overline{0} = \overline{0}$,
and thus $\lambda \wedge \mu$, $\lambda \wedge \rho$ are r-FI-separated sets with $\lambda = (\lambda \wedge \mu) \vee (\lambda \wedge \rho)$. But λ is r-FI-connected implies that $\lambda \wedge \mu = \overline{0}$ or $\lambda \wedge \rho = \overline{0}$.

(2) $\Rightarrow$ (3): If $\lambda \wedge \mu = \overline{0}$, $\lambda \leq \mu \vee \rho$ means that $\lambda = \lambda \wedge (\mu \vee \rho) = \lambda \wedge \rho$, and thus $\lambda \leq \rho$. Also, if $\lambda \wedge \rho = \overline{0}$, then $\lambda = \lambda \wedge \mu$, and then $\lambda \leq \mu$.

(3) $\Rightarrow$ (1): Let μ, ρ be r-FI-separated sets such that $\lambda = \mu \vee \rho$. Then, from (3), $\lambda \leq \mu$ or $\lambda \leq \rho$. If $\lambda \leq \mu$, then $\rho = (\mu \vee \rho) \wedge \rho = \lambda \wedge \rho \leq \mu \wedge \rho \leq \mathrm{cl}^*_\tau(\mu, r) \wedge \rho = \overline{0}$. Also, if $\lambda \leq \rho$, then $\mu = (\mu \vee \rho) \wedge \mu = \lambda \wedge \mu \leq \rho \wedge \mu \leq \mathrm{cl}^*_\tau(\rho, r) \wedge \mu = \overline{0}$. Hence, λ is r-FI-connected. □

Theorem 3. *Let $(X, \tau, \mathcal{I}), (Y, \sigma, \mathcal{I}')$ be fuzzy ideal topological spaces and $f : (X, \tau) \to (Y, \sigma, \mathcal{I}')$ is a mapping satisfying the condition* (IC). *Then, $f(\lambda) \in I^Y$ is r-FI-connected if $\lambda \in I^X$ is r-FI-connected.*

Proof. Let $\mu, \rho \in I^Y$ be r-FI-separated with $f(\lambda) = \mu \vee \rho$. That is, $\mathrm{cl}^*_\sigma(\mu, r) \wedge \rho = \mathrm{cl}^*_\sigma(\rho, r) \wedge \mu = \overline{0}$; $r \in I_0$. Then, $\lambda \leq f^{-1}(\mu) \vee f^{-1}(\rho)$, and from condition (IC), we get that

$$\begin{aligned}
\mathrm{cl}^*_\tau(f^{-1}(\mu), r) \wedge f^{-1}(\rho) &\leq \mathrm{cl}_\tau(f^{-1}(\mu), r) \wedge f^{-1}(\rho)\\
&\leq f^{-1}(\mathrm{cl}^*_\sigma(\mu, r)) \wedge f^{-1}(\rho)\\
&= f^{-1}(\mathrm{cl}^*_\sigma(\mu, r) \wedge \rho)\\
&= f^{-1}(\overline{0}) = \overline{0},
\end{aligned}$$

$$\begin{aligned}
\mathrm{cl}^*_\tau(f^{-1}(\rho), r) \wedge f^{-1}(\mu) &\leq \mathrm{cl}_\tau(f^{-1}(\rho), r) \wedge f^{-1}(\mu)\\
&\leq f^{-1}(\mathrm{cl}^*_\sigma(\rho, r)) \wedge f^{-1}(\mu)\\
&= f^{-1}(\mathrm{cl}^*_\sigma(\rho, r) \wedge \mu)\\
&= f^{-1}(\overline{0}) = \overline{0}.
\end{aligned}$$

Hence, $f^{-1}(\mu)$ and $f^{-1}(\rho)$ are r-FI-separated sets in X so that $\lambda \leq f^{-1}(\mu) \vee f^{-1}(\rho)$. But λ is r-FI-connected means, from (3) in Proposition 3.2, that $\lambda \leq f^{-1}(\mu)$ or $\lambda \leq f^{-1}(\rho)$, which means that $f(\lambda) \leq \mu$ or $f(\lambda) \leq \rho$. Thus, again from (3) in Proposition 3.2, we get that $f(\lambda)$ is r-FI-connected. □

Corollary 3. *If λ is r-fuzzy connected in (X, τ) or λ is r-FI-connected in $(X, \tau, \mathcal{I})$ with respect to a fuzzy ideal $\mathcal{I}$ on X, $f : (X, \tau) \to (Y, \sigma)$ is fuzzy continuous mapping, then $f(\lambda)$ is r-fuzzy connected in (Y, σ), and it is not necessary that $f(\lambda)$ is r-FI-connected with respect to a fuzzy ideal $\mathcal{I}'$ on Y. With condition* (IC), *$f(\lambda)$ is r-FI-connected whenever λ is r-fuzzy connected or λ is r-FI-connected. Moreover, $f(\lambda)$ is r-fuzzy connected whenever λ is r-fuzzy connected or λ is r-FI-connected.*

Proof. Clear from fuzzy continuity, (IC) and Theorem 3.1. □

The implications in the following diagram are satisfied whenever f satisfies condition (IC).

$$\begin{tikzcd}
\lambda\ is\ FI-conn. \arrow[r, "usual"] \arrow[d] \arrow[dr] & \lambda\ is\ F-conn. \arrow[d] \arrow[dl] \\
f(\lambda)\ is\ FI-conn. \arrow[r, "usual"'] & f(\lambda)\ is\ F-conn.
\end{tikzcd}$$

Only the implications in the following diagram are satisfied whenever f is fuzzy continuous.

$$\begin{tikzcd}
\lambda\ is\ FI-conn. \arrow[r, "usual"] \arrow[dr] & \lambda\ is\ F-conn. \arrow[d] \\
f(\lambda)\ is\ FI-conn. \arrow[r, "usual"'] & f(\lambda)\ is\ F-conn.
\end{tikzcd}$$

Proposition 8. *Any fuzzy point* $x_t, t \in I_0$ *is* r-*FI-connected, and consequently* $x_1\ \forall x \in X$ *is* r-*FI-connected.*

Proof. Clear. □

Definition 6. Let X be a non-empty set and $\lambda \in I^X$. Then, λ is r-FI-component if λ is maximal r-FI-connected set in X, that is, if $\mu \geq \lambda$ and μ is r-FI-connected set, then $\lambda = \mu$.

Proposition 9. *Let* $\lambda \neq \overline{0}$ *be* r-*FI-connected in X and* $\lambda \leq \mu \leq \mathrm{cl}^*_\tau(\lambda, r)$; $r \in I_0$. *Then,* μ *is* r-*FI-connected as well.*

Proof. Let ν, ρ be r-FI-separated sets in I^X such that $\mu = \nu \vee \rho$. That is, $\mathrm{cl}^*_\tau(\nu, r) \wedge \rho = \mathrm{cl}^*_\tau(\rho, r) \wedge \nu = \overline{0}$; $r \in I_0$. Since $\lambda \leq \mu$ implies that $\lambda \leq (\nu \vee \rho)$ and λ is r-FI-connected, then from (3) in Proposition 3.2, we have $\lambda \leq \nu$ or $\lambda \leq \rho$. From $\mu \leq \mathrm{cl}^*_\tau(\lambda, r)$ we get that

If $\lambda \leq \nu$, then $\rho = (\nu \vee \rho) \wedge \rho = \mu \wedge \rho \leq \mathrm{cl}^*_\tau(\lambda, r) \wedge \rho \leq \mathrm{cl}^*_\tau(\nu, r) \wedge \rho = \overline{0}$.
If $\lambda \leq \rho$, then $\nu = (\nu \vee \rho) \wedge \nu = \mu \wedge \nu \leq \mathrm{cl}^*_\tau(\lambda, r) \wedge \nu \leq \mathrm{cl}^*_\tau(\rho, r) \wedge \nu = \overline{0}$.
Hence, μ is r-FI-connected. □

4. Compactness in fuzzy ideal topological spaces

This section is devoted to introduce the notion of r-fuzzy ideal compact spaces.

Definition 7. Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space, $\lambda \in I^X, r \in I_0$. Then,

(1) λ is said to be r-fuzzy I-compact (r-FI-compact, for short) if for every family $\{\mu_j \in I^X : \tau(\mu_j) \geq r\ j \in J\}$ with $\lambda \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J such that
$$\mathcal{I}(\lambda\ \overline{\wedge}\ (\bigvee_{j \in J_0} \mu_j)) \geq r.$$

(2) λ is said to be r-fuzzy almost I-compact (r-FAI-compact, for short) if for every family $\{\mu_j \in I^X : \tau(\mu_j) \geq r\ j \in J\}$ with $\lambda \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J such that
$$\mathcal{I}(\lambda\ \overline{\wedge}\ (\bigvee_{j \in J_0} \mathrm{cl}^*_\tau(\mu_j, r))) \geq r.$$

(3) λ is said to be r-fuzzy nearly I-compact (r-FNI-compact for short) if for every family $\{\mu_j \in I^X : \tau(\mu_j) \geq r\ j \in J\}$ with $\lambda \leq \bigvee_{j \in J} \mu_j$, there exists a finite subset J_0 of J such that
$$\mathcal{I}(\lambda\ \overline{\wedge}\ (\bigvee_{j \in J_0} \mathrm{int}_\tau(\mathrm{cl}^*_\tau(\mu_j, r), r))) \geq r.$$

It is clear that: r-FI-compactness $\Longrightarrow$ r-FAI-compactness $\Longrightarrow$ r-FNI-compactness.

Remark 3. If $\mathcal{I} = \mathcal{I}^\circ$, then the concepts of:

(1) r-fuzzy compact and r-FI-compact are equivalent.
(2) r-fuzzy almost compact and r-FAI-compact are equivalent.
(3) r-fuzzy nearly compact and r-FNI-compact are equivalent.

Definition 8. Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space. Then, X is said to be r-fuzzy I-regular space if for each $\lambda \in I^X$, $r \in I_0$ with $\tau(\lambda) \geq r$,
$$\lambda = \bigvee_{j \in J} \{\lambda_j : \tau(\lambda_j) \geq r,\ \mathrm{cl}^*_\tau(\lambda_j, r) \leq \lambda\}.$$

It is clear that every r-fuzzy regular space is an r-fuzzy I-regular space. But if $\mathcal{I} = \mathcal{I}^\circ$, then the concept of r-fuzzy I-regular space and r-fuzzy regular space are equivalent.

Theorem 4. *Let* $(X, \tau, \mathcal{I})$ *be* r-*FAI-compact and* r-*fuzzy I-regular. Then, X is an* r-*FI-compact space.*

Proof. For every family $\{\mu_j \in I^X : \tau(\mu_j) \geq r,\ j \in J\}$ with $\overline{1} = \bigvee_{j \in J} \mu_j$. By r-fuzzy I-regularity of X, then for each $\tau(\mu_j) \geq r$, we have
$$\mu_j = \bigvee_{j_k \in J_K} \{\mu_{j_k} : \tau(\mu_{j_k}) \geq r,\ \mathrm{cl}^*_\tau(\mu_{j_k}, r) \leq \mu_j\}.$$

Hence, $\overline{1} = \bigvee_{j\in J}(\bigvee_{j_k\in J_K}\mu_{j_k})$. Since X is r-FAI-compact, then there exists a finite index subset $J_0 \times J_K$ of J such that

$$\mathcal{I}(\overline{1} \,\bar{\wedge}\, (\bigvee_{j\in J_0}(\bigvee_{j_k\in J_K} \mathrm{cl}^*_\tau(\mu_{j_k}, r)))) \geq r.$$

For each $j \in J_0$, $\bigvee_{j_k\in J_K} \mathrm{cl}^*_\tau(\mu_{j_k}, r) \leq \mu_j$, which implies that

$$\overline{1} \,\bar{\wedge}\, (\bigvee_{j\in J_0}(\bigvee_{j_k\in J_K} \mathrm{cl}^*_\tau(\mu_{j_k}, r))) \geq \overline{1} \,\bar{\wedge}\, (\bigvee_{j\in J_0}\mu_j).$$

Therefore, $\mathcal{I}(\overline{1} \,\bar{\wedge}\, (\bigvee_{j\in J_0}\mu_j)) \geq r$, and thus $(X, \tau, \mathcal{I})$ is r-FI-compact. □

Theorem 5. *Let $(X, \tau, \mathcal{I})$ be r-FNI-compact and r-fuzzy I-regular. Then, X is an r-FI-compact space.*

Proof. Similar to the proof of Theorem 4.1. □

Theorem 6. *Let $f : (X, \tau, \mathcal{I}_1) \to (Y, \sigma, \mathcal{I}_2)$ be injective fuzzy continuous mapping, $\lambda \in I^X$ is an r-FI-compact and $\mathcal{I}_1(\nu) \leq \mathcal{I}_2(f(\nu))\ \forall \nu \in I^X$. Then, $f(\lambda)$ is r-FI-compact as well.*

Proof. Let $\{\mu_j \in I^Y : \sigma(\mu_j) \geq r\ ,\ j \in J\}$ be a family with $f(\lambda) \leq \bigvee_{j\in J}\mu_j$. By fuzzy continuity of f, $\tau(f^{-1}(\mu_j)) \geq r$ and $\lambda \leq \bigvee_{j\in J} f^{-1}(\mu_j)$. By r-FI-compactness of λ, there exists a finite subset J_0 of J such that

$$\mathcal{I}_1(\lambda \,\bar{\wedge}\, (\bigvee_{j\in J_0}(f^{-1}(\mu_j)))) \geq r.$$

Since $\mathcal{I}_1(\nu) \leq \mathcal{I}_2(f(\nu))\ \forall \nu \in I^X$, then

$$\mathcal{I}_2[f(\lambda \,\bar{\wedge}\, (\bigvee_{j\in J_0}(f^{-1}(\mu_j))))] \geq r.$$

From f is injective, then $f(\lambda \,\bar{\wedge}\, (\bigvee_{j\in J_0}(f^{-1}(\mu_j)))) = f(\lambda) \,\bar{\wedge}\, (\bigvee_{j\in J_0}(\mu_j))$. Thus,

$$\mathcal{I}_2(f(\lambda) \,\bar{\wedge}\, (\bigvee_{j\in J_0}(\mu_j))) \ \geq\ r.$$

Hence, $f(\lambda)$ is r-FI-compact. □

The concept of a fuzzy operation, associated with a fuzzy topology τ, on a set X is a map $\alpha : I^X \times I_0 \to I^X$ so that $\mathrm{int}_\tau \leq \alpha \leq \mathrm{cl}_\tau$. This type of maps is called an expansion on X or a fuzzy operator on (X, τ). Let (X, τ_1) and (Y, τ_2) be two fuzzy topological spaces, α and β are fuzzy operators on X, θ and δ are fuzzy operators on Y, respectively [2].

Definition 9. Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space, α a fuzzy operator on X and $\lambda \in I^X,\ r \in I_0$. Then, λ is called r-fuzzy ideal α-compact (r-$FI\alpha$-compact for short) if for each family $\{\mu_j \in I^X : \tau(\mu_j) \geq r,\ j \in J\}$ with $\lambda \leq \bigvee_{j\in J}\mu_j$, there exists a finite subset J_0 of J such that

$$\mathcal{I}(\lambda \,\bar{\wedge}\, (\bigvee_{j\in J_0}\alpha(\mu_j, r))) \geq r.$$

It is clear that for $\alpha =$ identity operator (resp. $\alpha = \mathrm{cl}^*_\tau$ and $\alpha = \mathrm{int}_\tau \mathrm{cl}^*_\tau$), we get the r-FI-compact (resp. r-FAI-compact and r-FNI-compact).

Definition 10. [2] A mapping $f : (X, \tau, \mathcal{I}_1) \to (Y, \sigma, \mathcal{I}_2)$ is said to be fuzzy ideal $(\alpha, \beta, \theta, \delta, \mathcal{I})$-continuous if for every $\mu \in I^Y$, $\mathcal{I}_1[\alpha(f^{-1}(\delta(\mu, r)), r) \,\bar{\wedge}\, \beta(f^{-1}(\theta(\mu, r)), r)] \ \geq\ \sigma(\mu)$; $r \in I_0$, where α, β are fuzzy operators on X and θ, δ are fuzzy operators on Y.

(1) The concept of fuzzy almost ideal continuous ($FAIC$ for short) mapping is defined by: for every $\mu \in I^Y,\ r \in I_0$ with $\sigma(\mu) \geq r$, then

$$f^{-1}(\mu) \leq \mathrm{int}_\tau(f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}^*_\sigma(\mu, r), r)), r).$$

Here, $\alpha = id_X,\ \beta = \mathrm{int}_\tau,\ \delta = id_Y,\ \theta = \mathrm{int}_\sigma\,\mathrm{cl}^*_\sigma$ and $\mathcal{I}_1 = \mathcal{I}^\circ$.

(2) The concept of fuzzy weakly ideal continuous ($FWIC$ for short) mapping is defined by: for every $\mu \in I^Y,\ r \in I_0$ with $\sigma(\mu) \geq r$, then

$$f^{-1}(\mu) \leq \mathrm{int}_\tau(f^{-1}(\mathrm{cl}^*_\sigma(\mu, r)), r).$$

Here, $\alpha = id_X,\ \beta = \mathrm{int}_\tau,\ \delta = id_Y,\ \theta = \mathrm{cl}^*_\sigma$ and $\mathcal{I}_1 = \mathcal{I}^\circ$.

(3) The concept of fuzzy almost weakly ideal continuous ($FAWIC$ for short) mapping is defined by: for every $\mu \in I^Y,\ r \in I_0$ with $\sigma(\mu) \geq r$, then

$$f^{-1}(\mu) \leq \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{cl}^*_\sigma(\mu, r)), r), r).$$

Here, $\alpha = id_X,\ \beta = \mathrm{int}_\tau\,\mathrm{cl}_\tau,\ \delta = id_Y,\ \theta = \mathrm{cl}^*_\sigma$ and $\mathcal{I}_1 = \mathcal{I}^\circ$.

(4) The following diagram commutes.

$$\begin{tikzcd}
FAIC \arrow[r] \arrow[d] & FWIC \arrow[r] \arrow[d] & FAWIC \arrow[d] \\
FAC \arrow[r] & FWC \arrow[r] & FAWC
\end{tikzcd}$$

For $\mathcal{I}_2 = \mathcal{I}^{\circ}$, we get that:
$(FAC \Leftrightarrow FAIC)$, $(FWC \Leftrightarrow FWIC)$ and $(FAWC \Leftrightarrow FAWIC)$.

Theorem 7. *Let $(X, \tau, \mathcal{I}_1)$ and $(Y, \sigma, \mathcal{I}_2)$ be fuzzy ideal topological spaces, α a fuzzy operator on X and δ, θ are fuzzy operators on Y such that*

$$\nu \leq \alpha(\nu, r)\ \forall \nu \in I^X,\ \ \rho \leq \delta(\rho, r)\ \forall \rho \in I^Y,\ \ r \in I_0,$$

and $f : X \to Y$ is an injective mapping with $\alpha(f^{-1}(\delta(\rho, r)), r) \leq \mathrm{int}_\tau(f^{-1}(\theta(\rho, r)), r)$ $\forall \rho \in I^Y$, $r \in I_0$. If $\mu \in I^X$ is r-FI-compact and $\mathcal{I}_1(\mu) \leq \mathcal{I}_2(f(\mu))$. Then, $f(\mu) \in I^Y$ is r-FIθ-compact.

Proof. Let $\{\lambda_j \in I^Y : \sigma(\lambda_j) \geq r,\ \ j \in J\}$ be a family with $f(\mu) \leq \bigvee_{j\in J} \lambda_j$. Then, take $\mu_j = \mathrm{int}_\tau(f^{-1}(\theta(\lambda_j, r)), r)$ with $\tau(\mu_j) \geq r$ such that $\alpha(f^{-1}(\delta(\lambda_j, r)), r) \leq \mu_j \leq f^{-1}(\theta(\lambda_j, r))$. Also, since $f^{-1}(\delta(\lambda_j, r)) \leq \alpha(f^{-1}(\delta(\lambda_j, r)), r)$, $f^{-1}(\lambda_j) \leq f^{-1}(\delta(\lambda_j, r))$, then

$$f^{-1}(\lambda_j) \leq\ f^{-1}(\delta(\lambda_j, r)) \leq \mu_j \leq f^{-1}(\theta(\lambda_j, r)),$$

which means that

$$\mu \leq \bigvee_{j\in J} f^{-1}(\lambda_j) \leq \bigvee_{j\in J} \mu_j \leq f^{-1}(\bigvee_{j\in J} \theta(\lambda_j, r)),$$

that is, $\mu \leq \bigvee_{j\in J} \mu_j$ and μ is r-FI-compact. Then, there exists a finite set $J_0 \subseteq J$ such that

$$\mathcal{I}_1(\mu \barwedge (\bigvee_{j\in J_0} \mu_j)) \geq r.$$

From f is injective, then $f(\mu \barwedge (\bigvee_{j\in J_0} \mu_j)) = f(\mu) \barwedge (\bigvee_{j\in J_0} (f(\mu_j))) \ \geq\ f(\mu) \barwedge (\bigvee_{j\in J_0} \theta(\lambda_j, r))$. Thus,

$$\mathcal{I}_2(f(\lambda) \barwedge (\bigvee_{j\in J_0} (\theta(\mu_j, r)))) \ \geq\ r.$$

Hence, $f(\lambda)$ is r-$FI\theta$-compact. □

Corollary 4. *Let $(X, \tau, \mathcal{I}_1)$ and $(Y, \sigma, \mathcal{I}_2)$ be fuzzy ideal topological spaces. Let $f : X \to Y$ be an injective FWIC mapping, $\mathcal{I}_1(\nu) \leq \mathcal{I}_2(f(\nu))$ $\forall \nu \in I^X$, and $\mu \in I^X$ is an r-FI-compact. Then, $f(\mu) \in I^Y$ is an r-FAI-compact.*

Proof. Let $\alpha = id_X$, $\theta = \mathrm{cl}^*_\sigma$, $\delta = id_Y$, and $\mathcal{I} = \mathcal{I}^{\circ}$. Then, the result follows from Theorem 4.4. □

Corollary 5. *Let $(X, \tau, \mathcal{I}_1)$ and $(Y, \sigma, \mathcal{I}_2)$ be fuzzy ideal topological spaces. Let $f : X \to Y$ be an injective FAIC mapping, $\mathcal{I}_1(\nu) \leq \mathcal{I}_2(f(\nu))$ $\forall \nu \in I^X$, and $\mu \in I^X$ is an r-FI-compact. Then, $f(\mu) \in I^Y$ is an r-FNI-compact.*

Proof. Let $\alpha = id_X, \theta = \mathrm{int}_\sigma\, \mathrm{cl}^*_\sigma, \delta = id_Y$, and $\mathcal{I} = \mathcal{I}^{\circ}$. Then, the result follows from Theorem 4.4. □

5. Conclusion

Joining the concept of fuzzy ideal to the concept of fuzzy topology on an underlying set X has some effects as we shown in the paper. Fuzzy separation axioms, fuzzy connectedness and fuzzy compactness defined in a fuzzy ideal topological space were different from those defined in a fuzzy topological space. Although the concept of fuzzy ideal is independent from the concept of fuzzy topology, but studying the fuzzy ideal topological spaces added some results new and different.

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